1.1 Granulometries or Pattern Spectra- of Binary Images

In this section we consider filtering a binary image with a sequence of structuring elements of increasing size. This will have a sieving effect of removing larger and larger objects from the output image. Let S be a binary image and B a structuring element. The method of granulometry is useful for size and shape analysis of granular images [Dougherty, 1992, pp. 76; Margos, 1989]. It is a morphological method for characterizing granular images by means of the manner in which they are sieved through various shapes and sizes of structuring elements. Besides application to grains (particles), the method is useful for texture and shape analysis.

In this section we consider filtering a binary image with a sequence of structuring elements of increasing size. This will have a sieving effect of removing larger and larger objects from the output image. Let S be a binary image. If B_1, B_2, B_3, ... is an increasing sequence of structuring elements, then the filtered images form a decreasing sequence

$S \circ B_1 > S \circ B_2 > S \circ B_3 > \ldots\ldots\ldots$

Here B is a structuring element and $nB = B \oplus B \oplus \cdots \oplus B$ performed n times. If B contains the origin, then nB > B. Consider $S \circ nB$ as a filter. If n>m then $S \circ nB \subseteq S \circ mB$ since $nB \supseteq mB$. If A(S) is the area of set S, then $A(S \circ nB) \le A(S \circ mB)$. Recall that $S \circ nB = (S \Theta nB) \oplus nB$ and $S \bullet nB = (S \oplus nB) \Theta nB$. With S, B fixed and n varying $A(n) = A(S \circ nB)$ is a decreasing function of n. Let A(0) be the area of S. Then A(0) ≥ A(1) ≥ A(2) ≥ .... ≥ A(n). Let $ps(k) = \frac{dF}{dk}$. Let $F(n) = 1 - \frac{A(n)}{A(0)}$ and $0 \le F(n) \le 1$. F is a cumulative probability function. The quantity ps(k) is called a granularity size distribution or pattern spectrum. The following equations hold.

$$\frac{dF}{dk} = F(k+1) - F(k) = \left[1 - \frac{A(k+1)}{A(0)}\right] - \left[1 - \frac{A(k)}{A(0)}\right]$$

$$\frac{dF}{dk} = \frac{A(k) - A(k+1)}{A(0)}$$

$$ps(k) = \frac{dF}{dk}$$

$$ps(k) = \frac{A(k) - A(k+1)}{A(0)} = \frac{A(S \circ kB) - A(S \circ (k+1)B)}{A(S)}$$

The p(k) term gives the normalized area of the features that pass $S \circ kB$ but does not pass $S \circ (k+1)B$, These are the features of size kB. The following example demonstrates these calculations.

example

0	0	0	0	0	0	0	0	
0	0	0	1	1	0	1	0	
0	0	1	1	1	0	0	0	
0	1	1	1	1	1	0	0	*S*
0	0	1	1	1	0	0	0	
0	0	0	1	0	1	1	0	
0	0	0	1	0	1	1	1	
0	0	0	0	0	0	1	0	

	1	
1	①	1
	1	

B

		1		
	1	1	1	
1	1	①	1	1
	1	1	1	
		1		

2B

Figure 1. Example of Pattern Spectra

0	0	0	0	0	0	0	0	
0	0	0	1	0	0	0	0	
0	0	1	1	1	0	0	0	
0	1	1	1	1	1	0	0	
0	0	1	1	1	0	0	0	$S \circ B$
0	0	0	1	0	0	1	0	
0	0	0	0	0	1	1	1	
0	0	0	0	0	0	1	0	

0	0	0	0	0	0	0	0	
0	0	0	1	0	0	0	0	
0	0	1	1	1	0	0	0	
0	1	1	1	1	1	0	0	
0	0	1	1	1	0	0	0	$S \circ 2B$
0	0	0	1	0	0	0	0	
0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	

Figure 2. Example Pattern Spectra Calculations

For the above example,

A(0)=22
A(1)=18
A(2)=13
A(3)=0

$$F(0) = 1 - \frac{A(0)}{A(0)} = 0$$

$$F(1) = 1 - \frac{A(1)}{A(0)} = 1 - \frac{18}{22} = \frac{4}{22}$$

$$F(2) = 1 - \frac{A(2)}{A(0)} = 1 - \frac{13}{22} = \frac{9}{22}$$

$$F(3) = 1 - \frac{A(3)}{A(0)} = 1 - 0 = 1$$

F(4)=1.
ps(0)=4/22 this is the area of elements of size < B
ps(1)=9/22-4/22=5/22 this is the area of elements of size B
ps(2) = 1 - 9/22=13/22 this is the area of elements of size 2B.
ps(3) = 1-1=0 there are no elements of this size.

1.1.1 Object Shape

The pattern spectra also gives information about the shape of an object. Figure 2 shows an image with a square pattern of 1's in a background of 0's. Table 1 shows the pattern spectrum with respect to the three different structuring elements in Figure 2. Figure 3 shows the corresponding plots. This example illustrates useful information that can be obtained from the pattern spectra [Sundararaman, 1994, pp. 90]. The pattern spectrum obtained from the opening of an image S with a structuring element B, can yield the *B*-shapeness of S [Maragos, 1989]. If the pattern spectrum for the opening of S with B is given by ps(1), ps(2),....., ps(i) then for a particular i = n, ps(i) = 0, for all i>n, if S and B are fixed. Then the B-shapeness of S can be determined by ps(n)/A(S), where A(S) is the area of S.

The following figure shows an example of these calculations for a square and a cross structuring element.

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

Example Image

1	1	1
1	1	1
1	1	1

B square

	1	
1	1	1
	1	

C cross

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

2B

		1		
	1	1	1	
1	1	1	1	1
	1	1	1	
		1		

2C

0	1	1	1	1	0
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
0	1	1	1	1	0

$A \circ C$

0	0	1	1	0	0
0	1	1	1	1	0
1	1	1	1	1	1
1	1	1	1	1	1
0	1	1	1	1	0
0	0	1	1	0	0

$A \circ 2C$

Figure 3. Example shape calculation

The following table computes the shape measures. The square structuring element has the maximum B-shapeness, indicating that the given image is more like a square than a cross.

Table 1 Area, ps(n), for Increasing Structuring Element Size

Structuring Element Sequence	0	1	2	3
Square	36	36	36	0
Cross	36	32	24	0

Table 2. Area Difference and Shapeness

	0	1	2	3	B-Shapeness
Square	0	0	0	0	1.00
Cross	0	4	12	24	.67

The concept of local granulometric size distribution or the pattern spectrum can be used to segment images based upon local texture. Pixel classification based on texture requires that a pixel be classified according to the nature of the surrounding image region. Specifically, a parametric texture feature at a pixel is a measure based on the image values (for e.g., in the binary case this could be the number of 1's) in some window about the pixel itself. In such an approach, the image (binary) is successively opened and at each stage an image pixel count is taken. To measure image texture local to a given pixel, the count in the window about the pixel is taken resulting in a local granulometric size distribution about the pixel. Normalization yields a local pattern spectrum density at each pixel, and each of these possesses moments. The moments across a sub region of an image possessing uniform texture is likely to remain stable throughout the sub region. Thus differing sub regions with different but uniform textures can be differentiated based on the local pattern spectrum moments. For example if the local pattern spectrum mean (μ_{ps}) is considered, then each pixel p will have a $\mu_{ps,p}$ and the pixel regions could be segmented based on the local $\mu_{ps,p}$ values [Dougherty, 1992].

1.1.2 Foreground/Background Measurement in a Window.

Consider constructing a measurement of the content an object occupies in a window. This calculation follows from the calculation of the pattern spectrum. If A_w is the area of the window then one can obtain the percentage area of objects by computing $dA = ps(k) * A(0) = A(k) - A(k+1)$ that gives the area of the features of size kB. We are measuring the area of size kB area of the foreground of the image. One can divide by A_w to obtain the percentage area of the objects [Sundararaman, 1994].

Consider the image shown in figure 5., and the binary opening of this image with the sequence of structuring elements in Figure 6. For the openings of the image in Figure 6, the number of 1's in the image, 'A', after each opening is given in Table 3., and the count difference dA = $(A_{n+1} - A_n)$, in Table 4. Here we are measuring the area of the foreground in the image, with each opening. The count difference gives a measure of 'dA' or the change in area with each successive opening. From Table 4., it can be seen that as the size of the structuring element increases from (7x7) to (9x9), the foreground (or the 1's) is completely covered up and it results in a count difference, or 'dA', equal to the foreground area. This indicates that the foreground has an area equal to the size of the (7x7) structuring element, i.e., the foreground area is equal to 49. The sum of the dA's for the openings, sum(dA), is equal to the foreground area. Table 4 also gives the value of sum(dA).

```
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 1 1 1 1 1 1 1 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0
```

Figure 4. A Binary Image

```
①

1 1 1
1 ① 1
1 1 1

1 1 1 1 1
1 1 1 1 1
1 1 ① 1 1
1 1 1 1 1
1 1 1 1 1

1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 ① 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 ① 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
```

① : The Origin of the structuring element

Figure 5. A Sequence of Increasing Structuring Elements

	Increasing Structuring Elements					
	1 [1x1]	**2 [3x3]**	**3 [5x5]**	**4 [7x7]**	**5 [9x9]**	**6 [11x11]**
Count A	49	49	49	49	0	0

Table 3. Counts,ps(n). of the Openings with the Structuring Elements

	1	**2**	**3**	**4**	**5**	**Sum of dA**
Count Difference, dA	0	0	0	49	0	49

Table 4. Count Differences for the Structuring Elements.

In the following example we have three different binary images varying in the number of 1's dispersed in a window background of 0's. The pattern spectra for the binary opening of the three images, "imga", "imgb", and "imgc" with the structuring element sequence is given in Table 5. The values of ∑dA and the corresponding percentages relative to the window size are also given in Table 6. Note that for imgc it has foreground content of 71 of which 15 come from a 3 by 3 size structuring element and the rest from a 1 by 1 structuring element. These numbers indicate the percentage foreground contribution from each size element.

0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	1	0	0	1	0
0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	0	1	0	0	0	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	1	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

Figure 6. Imga

0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	1	0	1	1	0	1	0
0	0	0	1	1	0	0	0	0	1	1	0
0	0	0	0	0	1	1	0	0	0	0	0
0	1	1	0	0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0	0	1	1	0
0	0	1	1	0	1	1	0	1	1	0	0
0	0	0	0	0	0	1	1	0	1	1	0
0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	0	1	1	0	0
0	0	0	0	1	1	0	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

Figure 7. Imgb

0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	1	1	1	1	0
0	0	1	1	1	1	0	1	1	1	1	0
0	1	0	1	0	1	1	1	0	1	0	0
0	1	1	0	0	0	1	1	1	1	0	0
0	0	0	1	1	1	1	1	0	1	1	0
0	0	1	1	1	1	1	0	1	1	0	0
0	1	1	1	0	0	1	1	1	1	1	0
0	0	1	1	1	0	1	1	1	1	0	0
0	0	1	1	1	0	1	1	1	1	0	0
0	0	0	1	1	1	0	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0

Figure 8 . Imgc

①

1 1 1
1 ① 1
1 1 1

1 1 1 1 1
1 1 1 1 1
1 1 ① 1 1
1 1 1 1 1
1 1 1 1 1

1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 ① 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 ① 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1

① : The Origin of the structuring element

Figure 9. Increasing Structuring Elements

Table 5. Counts for imga, imgb, imgc for the Structuring Elements

	Count, A				
	1 [1x1]	**2 [3x3]**	**3 [5x5]**	**4 [7x7]**	**5 [9x9]**
imga	20	0	0	0	0
imgb	42	0	0	0	0
imgc	71	15	0	0	0

Table 6 Count Difference

	Count Difference, dA					
	1	**2**	**3**	**4**	**Sum of dA**	**Sum of dA%**
imga	20	0	0	0	20	14%
imgb	42	0	0	0	42	29%
imgc	56	15	0	0	71	49%

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