1. Gray-Scale Morphology

The concepts from morphology can be extended to the case of gray-scale images. The following development has substantial parts from the reference [Dougherty, 1992, pp. 91]. First let define some terminology. Let f be a function. Let the domain of f be defined by $\text{Dom}[f] = \{\text{u} \mid \text{f(u) is defined}\}$. Now f translated by pixel p is $f_p(u) = f(u - p)$. This is translation to the right. Now consider vertical translation of a function. Vertical translation by a scalar is (f+a)(p)=f(p)+a where a is a scalar. Morphological translation is then defined to be $(f_p + a)(u) = f(u - p) + a$. If f and g are functions, then g is beneath f , g<<f, if the

a. Dom[g] is contained in Dom[f] and

b. $\text{g(p)} \le \text{f(p)} \;\; \forall \text{p in Dom[g]}$.

Often one may places the value $-\infty$ for function values outside its domain of definition.

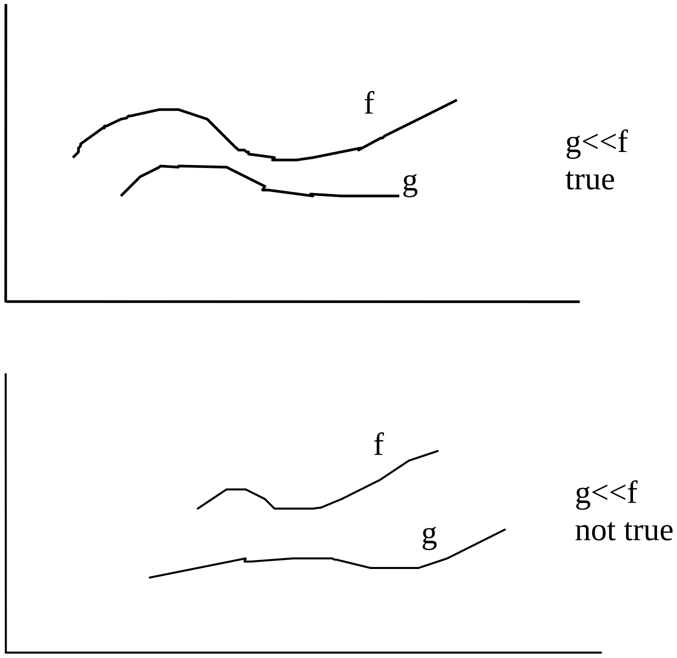


Figure 1. Function Beneath Another Function

Let us now define the minimum function of two functions. The minimum of the functions f and g is denoted by $f \wedge g$. It is defined in the following manner

If p is in $Dom[f] \cap Dom[g]$ then $(f \wedge g)(p) = \min[f(p), g(p)]$

else = $-\infty$ Instead of the $-\infty$ one might just let it be not defined.

Let us now define the maximum of two functions. The maximum of the functions f and g is denoted by $f \vee g$. It is defined in the following manner If p is in $Dom[f] \cap Dom[g]$ then $(f \vee g)(p) = \max[f(p), g(p)]$. If p is in $Dom[f] \cap Dom[g]^c$ then $(f \vee g)(p) = f(p)$. If p is in $Dom[f]^c \cap Dom[g]$ then $(f \vee g)(p) = g(p)$. Otherwise it is not defined.

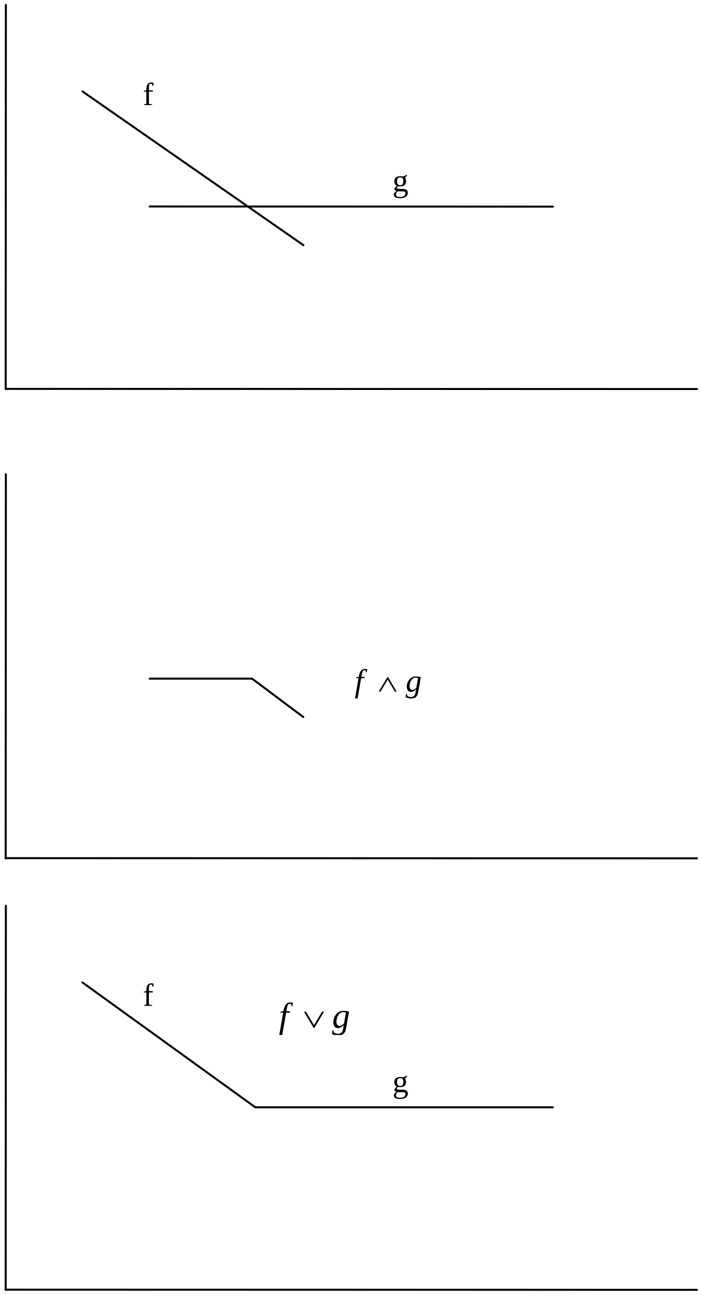


Figure 2. Minimum of Functions

The reflection of a function is defined by $\breve{h}(p) = -h(-p)$.

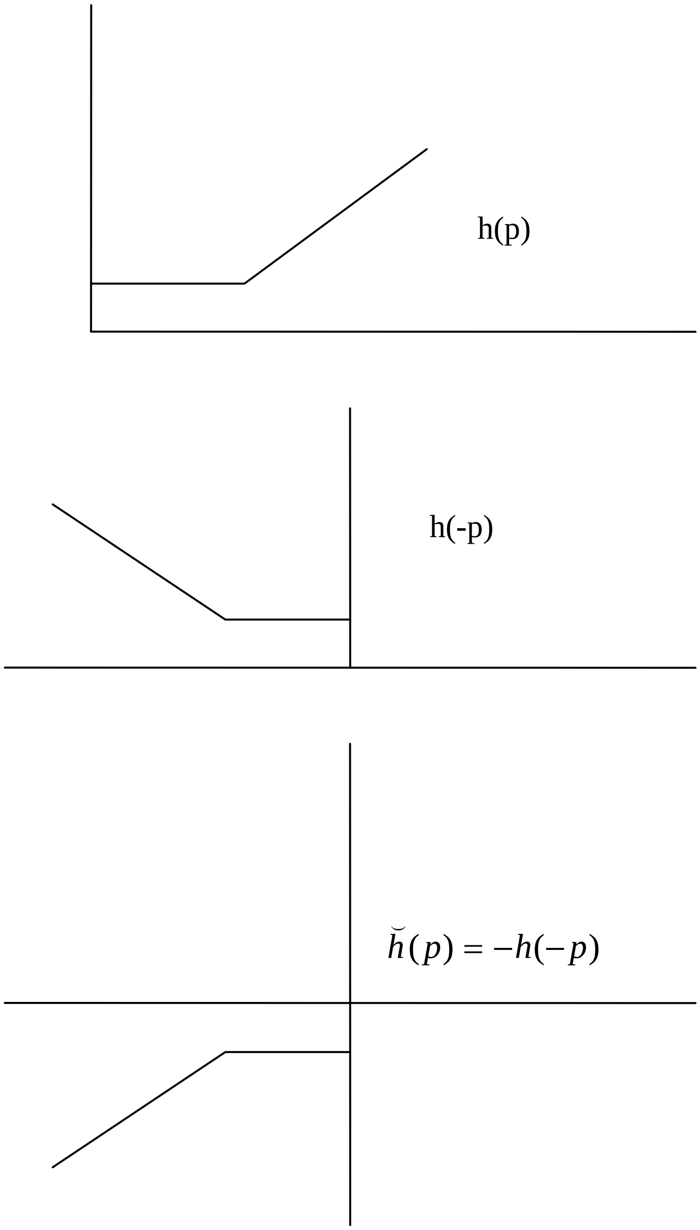


Figure 3. Reflection of a Function

1.1 Erosion

We can now define gray-scale erosion. Gray-scale erosion is defined in terms of an image function g and a function f which serves as the structuring element [Dougherty, 1992, pp. 95; Gonzalez and Woods, 1992, pp. 550]. The definition is $(g\Theta f)(p) = \max\{a | f_p + a << g\}$. That is f is translated by p, and it must be true that $Dom[f_p] \subseteq Dom[g]$, and $f_p(u) + a \leq g(u) \forall u \in Dom[f_p]$. The domain of the translated function f_p must be a subset of the domain of g. We slide the structuring element f until its origin lies on p and then we find a such that a is the maximum amount we can push f up by a and still have it remain under g. Erosion reduces the peaks and enlarges the values of a function.

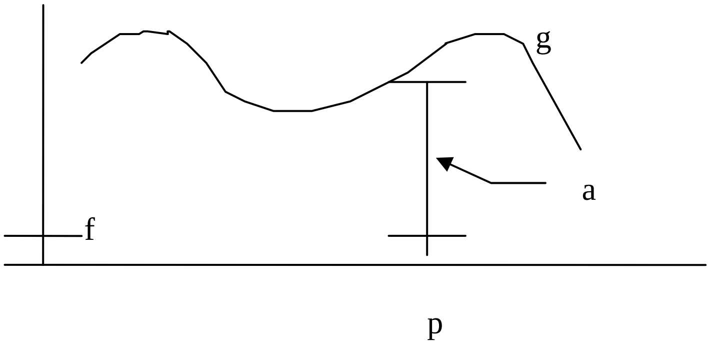


Figure 4. Erosion of Functions

There are other alternative definitions.
If $Dom[f_p] \subseteq Dom[g]$ then
$(g\Theta f)(p) = \min\{g(u) - f_p(u) \mid u \in Dom[f_p]\}$. This last equation is the same equation as $(g\Theta f)(p) = \min\{g(u) - f(u - p) \mid u \in Dom[f_p]\}$.

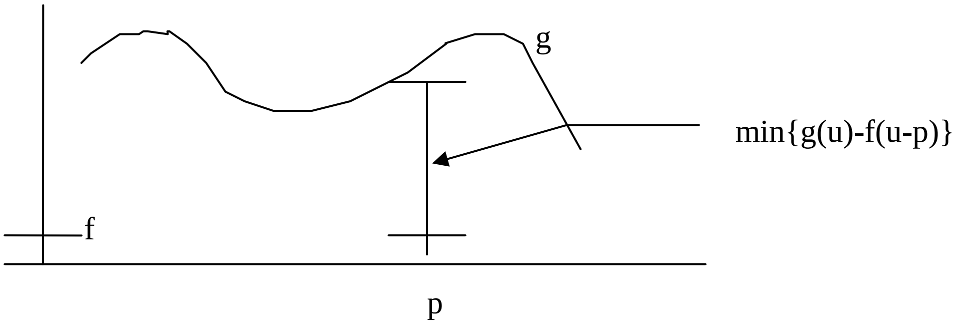


Figure 5. Erosion of Functions

Another equation is given below. If $\forall u \in Dom[f]$ it is true that $(p + u) \in Dom[g]$ then
$(g\Theta f)(p) = \min\{g(p + u) - f(u)\}$.

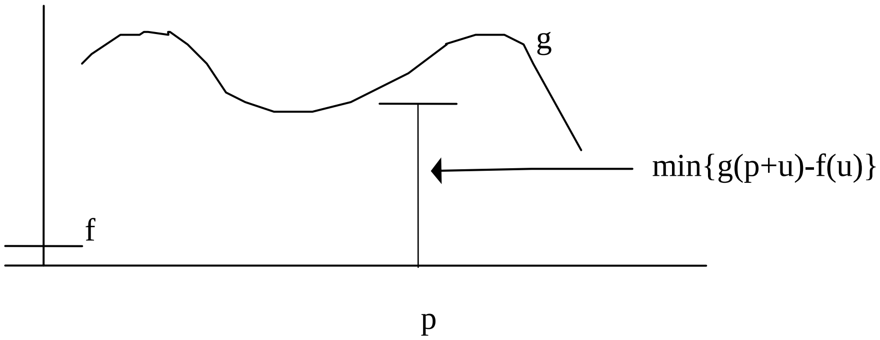


Figure 6. Erosion of Functions

The erosion operator with a flat structuring element reduces the peaks and enlarges the valleys of a function.

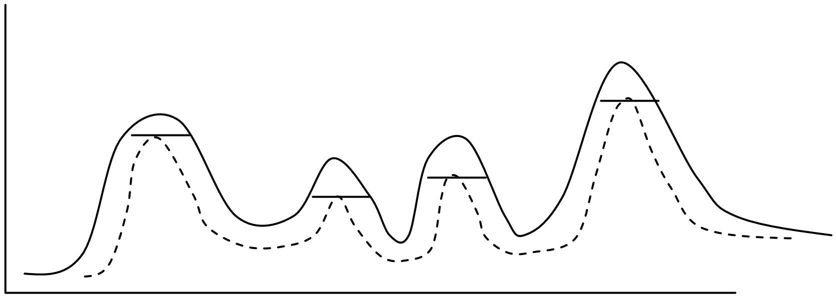


Figure 7. Erosion with Flat Function

Consider another example where the structuring function f is not constant

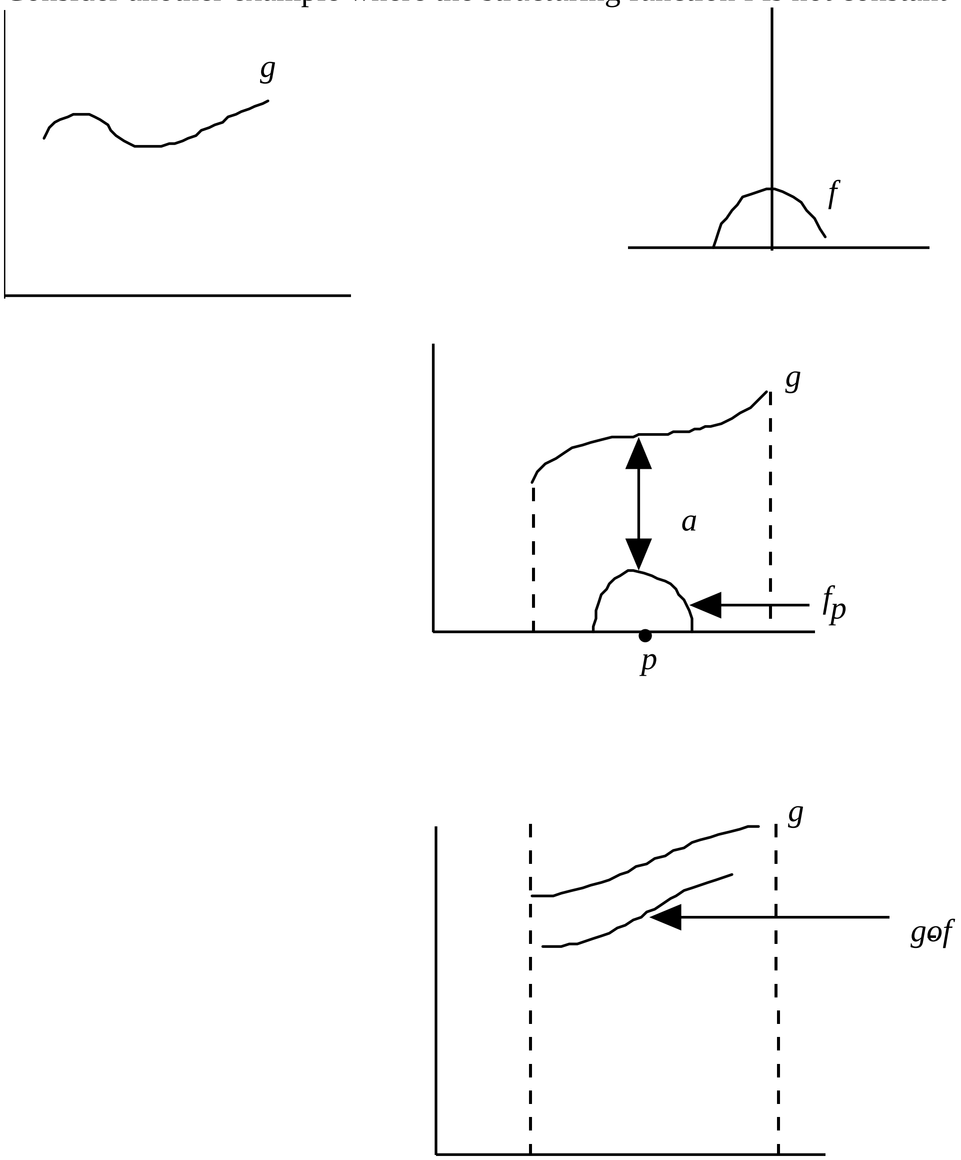


Figure 8. Erosion with non-constant Structuring Functions

Some examples of the calculation are shown in the following examples.

The first examples has the following function g and structuring function f. The structuring function is a constant function.

g=(7, 9, 8, 3, 8, 9, 9)
f=(0,⓪, 0)
The result of erosion is
$(g\Theta f)$ =(*,7,3,3,3,8,*).
Note that the domain of the erosion is shrunk similar to the case of binary erosion.

Consider another example where the structuring function is not constant.
g = (7, 9, 8, 3, 8, 9, 9)
f = (-3, ⓪, -3).
The result of erosion is
$(g\Theta f)$ = (*, 9, 6, 3, 6, 9, *).

Consider now the case of flat structuring elements. In this case, f is a constant. One can usually assume f is zero on its domain of definition otherwise the result differs by a constant offset. Consider erosion.

$(g\Theta f)(p) = \min\{g(u) - f_p(u) \mid u \in Dom[f_p]\}$

Let D be Dom[f] then D+p is $\mathrm{Dom}[\mathrm{f}_\mathrm{p}]$ and recall that $f_p(u) = f(u-p)$. Therefore,

$(\mathrm{g}\Theta\mathrm{f})(\mathrm{p}) = \min\left\{\mathrm{g}(\mathrm{u}) \mid \mathrm{u} \in \mathrm{D} + \mathrm{p}\right\}$

which is a min filter. Also note that

$(\mathrm{g}\Theta\mathrm{f})(\mathrm{p}) = \min\left\{\mathrm{g}(\mathrm{p}+\mathrm{u})\text{-}\mathrm{f}(\mathrm{u}) \mid \mathrm{u} \in \mathrm{Dom}[\mathrm{f}], \mathrm{u}+\mathrm{p} \in \mathrm{Dom}[\mathrm{g}]\right\}$ and when f(u)=0 on its domain. And

$(\mathrm{g}\Theta\mathrm{f})(\mathrm{p}) = \min\left\{\mathrm{g}(\mathrm{p}+\mathrm{u}) \mid \mathrm{u} \in \mathrm{Dom}[\mathrm{f}], \mathrm{u}+\mathrm{p} \in \mathrm{Dom}[\mathrm{g}]\right\}$.

1.2 Dilation

Now let us define gray-scale dilation element [Dougherty, 1992, pp. 100; Gonzalez and Woods, 1992, pp. 549]. Gray-scale dilation is defined by $(g \oplus f)(p) = \max\{g_u(p) + f(u) \mid u \in Dom[f], \quad p - u \in Dom[g]\}$. This may also be stated as

$(g \oplus f)(p) = \max\{g(p-u) + f(u) \mid u \in Dom[f], p-u \in Dom[g]\}$.

Recall that $g = -\infty$ outside of its domain of definition

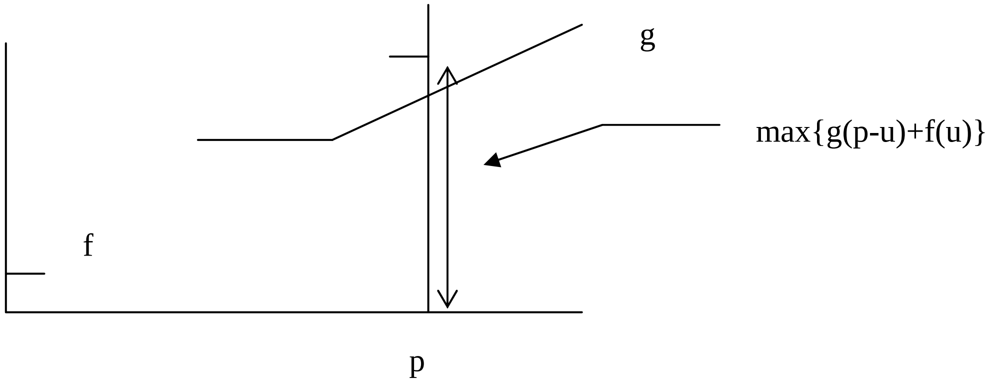


Figure 9. Dilation of Functions

The following figure shows how dilation operates.

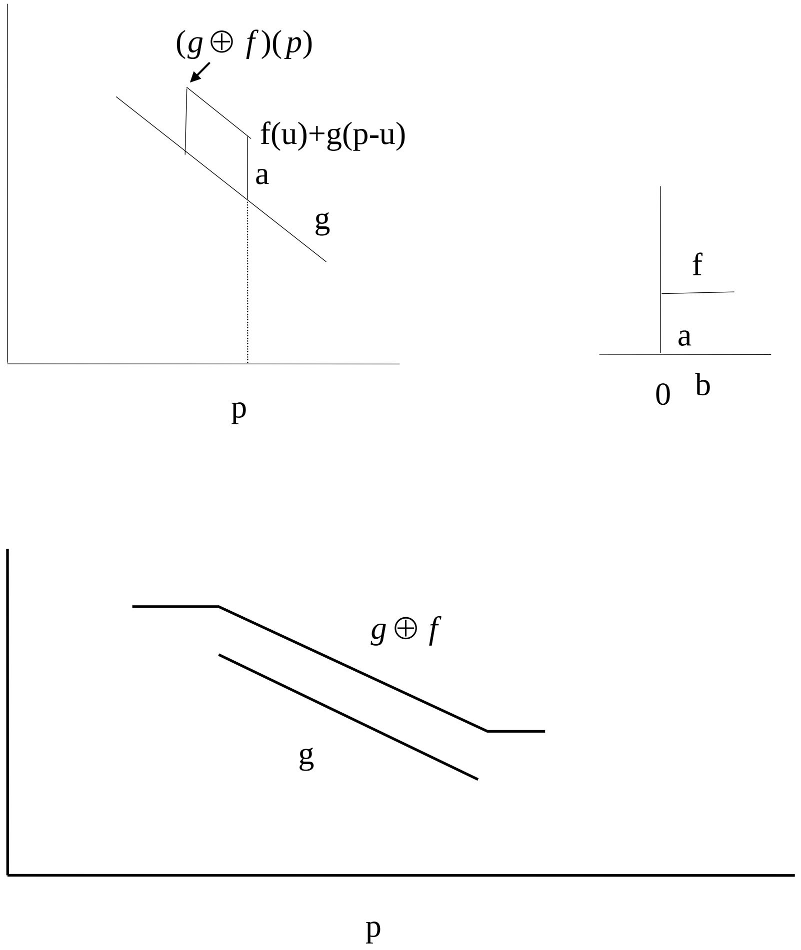


Figure 10. Dilation of Functions

An alternative definition for dilation is defined by $(g \oplus f)(p) = \min\{a \mid \breve{f}_p + a >> g\}$. We take the reflection of the structuring function f, translate it to p, and find the mininum it can be pushed up and lie above g when g is limited to the domain of the translated structuring function element [Dougherty, 1992, pp. 100; Gonzalez and Woods, 1992, pp. 549]. Note that the domain of the function is increased by the dilation operation similar to the binary case. The dilation operator fills the valleys and enlarges the peaks of a function.

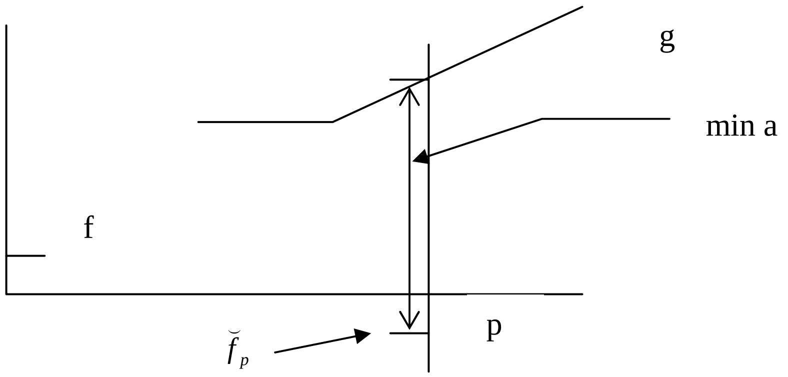


Figure 11. Dilation of Functions

An alternative definition is
$(g \oplus f)(p) = \max\{g_u(p) + f(u) \mid u \in Dom[f], \quad p - u \in Dom[g]\}$. This may also be stated as
$(g \oplus f)(p) = \max\{g(p-u) + f(u) \mid u \in Dom[f], p-u \in Dom[g]\}$.
Recall that $g = -\infty$ outside of its domain of definition

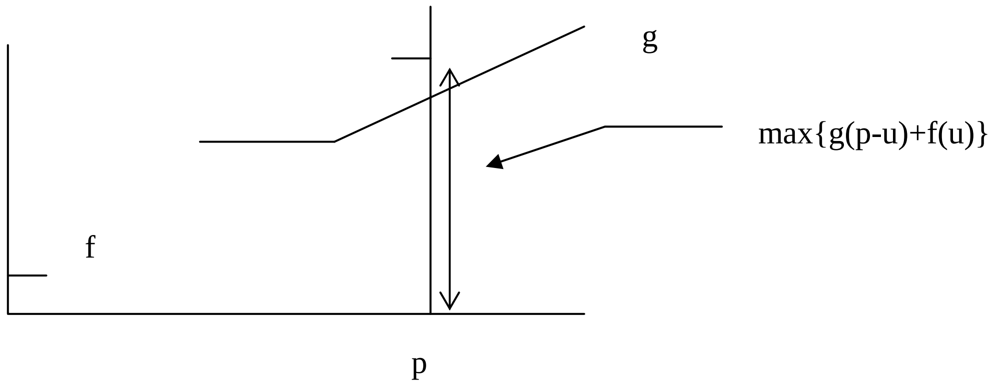


Figure 12. Dilation of Functions

The following figure shows how dilation operates.

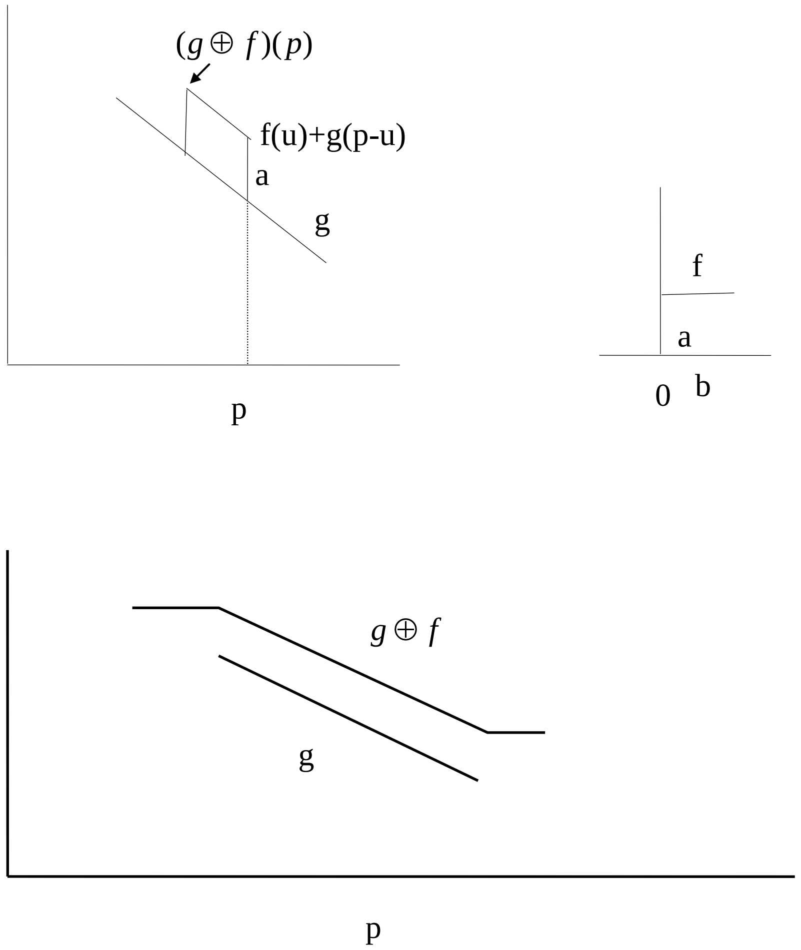


Figure 13. Dilation of Functions

The following figure shows dilation for a non-constant structuring function f.

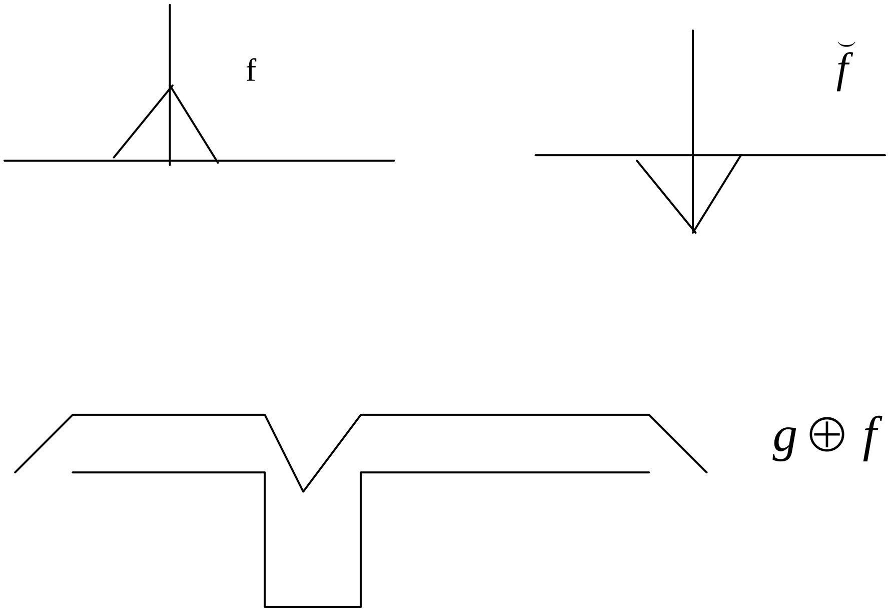


Figure 14. Dilation with non-constant Structuring Function

Consider the following example dilation calculation [Dougherty, 1992, pp. 100].

g=(7, 9, 8, 3, 8, 9 , 9)
f=(-3, 0, -3)

		7	9	8	3	8	9	9		
-3	0	-3						-3	0	-3
	-3	0	-3				-3	0	-3	
		-3	0	-3		-3	0	-3		
			-3	0	-3					
				-3	0	-3				
					-3	0	-3			
	4	7	9	8	5	8	9	9	6	

Figure 15. Example Dilation Calculation

Consider two points in detail. The first is the first point with value 8.

	7	9	8	3	8	9	9
		9	8	3			
		-3	0	-3			
max of the sums		6	8	0			
ans			8				

The second point is the point with the 7.

		7	9	8	3	8	9	9
		7	9	8				
	-3	0	-3					
max of the sums		7	6					
ans		7						

The following is another example of a dilation calculation.

f			1	2	③	2	1						
g			5	5	1	1	1	1	1	5	5		
$(g \oplus f)$	6	7	8	8	7	6	4	6	7	8	8	7	6

Figure 16. Example Dilation Calculation

Consider now the case of flat structuring elements. In this case, *f* is a constant. One can usually assume *f* is zero on its domain of definition otherwise the result differs by a constant offset. The equation for dilation is

$$(g \oplus f)(p) = \max\{g(p-u) + f(u) \mid u \in Dom[f], p-u \in Dom[g]\}$$

which becomes $(g \oplus f)(p) = \max\{g(p-u) \mid u \in Dom[f], p-u \in Dom[g]\}$.

This is a maximum filter.

1.3 Properties of erosion and dilation.

The following are properties of erosion and dilation [Dougherty, 1992, pp. 103].

$$(f \Theta g) \Theta h = f \Theta (g \oplus h)$$

$$f \oplus (g \vee h) = (f \oplus g) \vee (f \oplus h)$$ $\vee$ max

$$f \Theta (g \vee h) = (f \Theta q) \wedge (f \Theta h)$$ $\wedge$ min

$$(f \wedge g) \Theta h = (f \Theta h) \wedge (g \Theta h)$$

Duality property.

$$q \oplus f = -\left[(-g) \Theta \left(-\overset{\cup}{f}\right)\right]$$

here negation of $-\infty$ is $+\infty$

Another alternative is when a function is undefined the negation is also undefined in this case the equation is valid in

$$Dom\left[q \Theta \overset{\cup}{f}\right]$$

For flat structuring elements with zero values

$(g\Theta f)(p) = \min\left\{g(p+u) - f(u) \mid u \in Dom[f], u + p \in Dom[g]\right\}$ which becomes

$(g\Theta f)(p) = \min\left\{g(p+u) \mid u \in Dom[f], u + p \in Dom[g]\right\}$.

$(g \oplus f)(p) = \max\left\{g(p-u) + f(u) \mid u \in Dom[f]\right\}$ which becomes

$(g \oplus f)(p) = \max\left\{g(p-u) \mid u \in Dom[f]\right\}$.

These filters are related to rank filters.

1.4 Opening

Opening if defined from erosion and dilation in a similar manner to the binary case [Dougherty, 1992, pp. 111].

$g \circ f = (g \Theta f) \oplus f$

An alternative definition is $g \circ f = \max\{ f_p + a \mid f_p + a << g \}$. This may be stated as f is translated by p and $f_p(u) + a \le g(u) \; \forall \; u \in Dom[f_p]$ i.e. $f(u - p) + a \le g(u)$.

In this case of opening one slides the structuring element beneath the image. At each pixel the value is the maximum "a" such that the structuring element when displaced vertically by "a" still fits beneath the image.

1.5 Closing

The closing operation is defined by

$g \bullet f = (g \oplus f) \Theta f$ [Dougherty, 1992, pp. 113; Gonzalez and Woods, 1992, pp. 552].

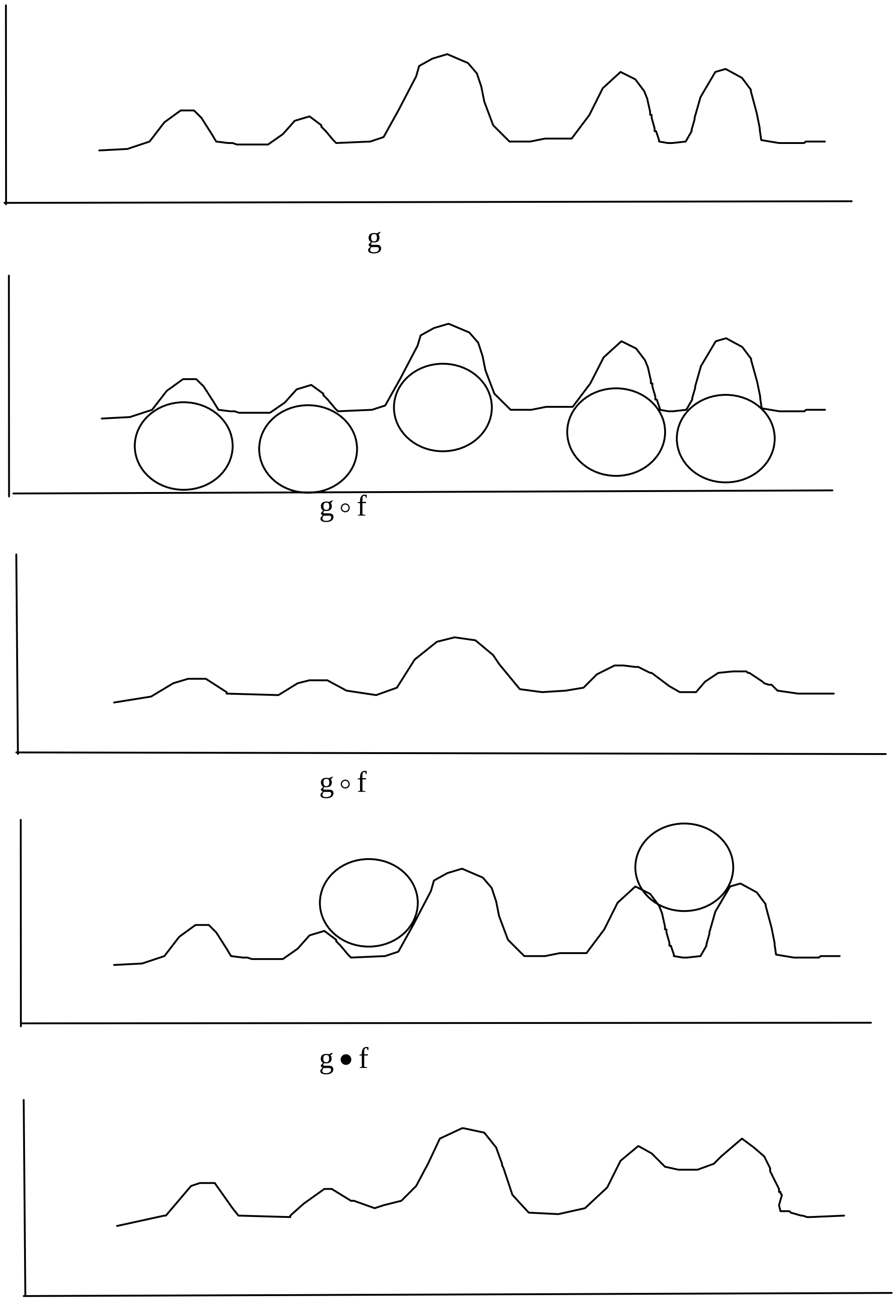
There is a duality between opening and closing. If one defines

$g^c = -g$ and

Duality $g \bullet f = g^c \circ \breve{f}$.

$g \bullet f = -[(-g) \circ (-f)]$

Closing may be viewed as sliding the structuring element over the image. At each pixel the value is the minimum "a" such that the structuring element displaced vertically by "a" still fits above the image [Gonzalez and Woods, 1992, pp. 554].



g

$g \circ f$

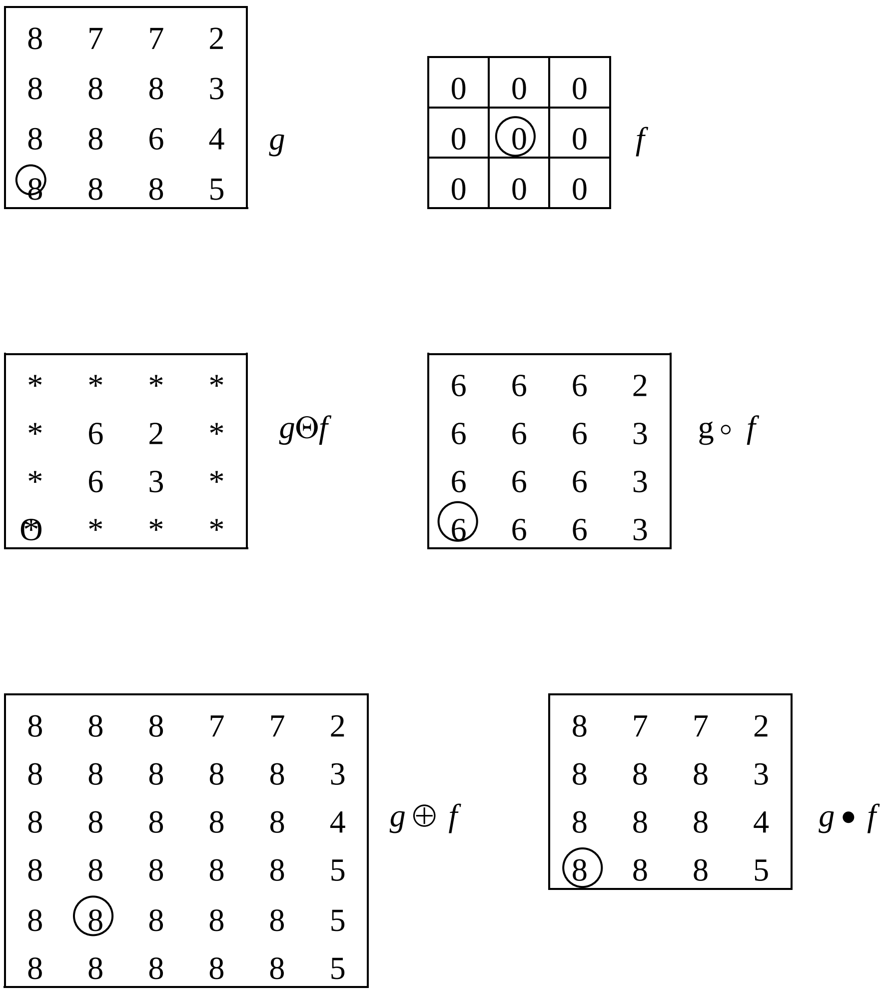
$g \circ f$

$g \bullet f$

$g \bullet f$

Figure 17. Dilation Operation

Consider the following numerical examples.



8	7	7	2
8	8	8	3
8	8	6	4
8	8	8	5

g

0	0	0
0	0	0
0	0	0

f

*	*	*	*
*	6	2	*
*	6	3	*
*	*	*	*

$g \Theta f$

6	6	6	2
6	6	6	3
6	6	6	3
6	6	6	3

$g \circ f$

8	8	8	7	7	2
8	8	8	8	8	3
8	8	8	8	8	4
8	8	8	8	8	5
8	8	8	8	8	5
8	8	8	8	8	5

$g \oplus f$

8	7	7	2
8	8	8	3
8	8	8	4
8	8	8	5

$g \bullet f$

Figure 18. Example Dilation Calculation

The next example shows a non-constant structuring function.

8	7	7	2
8	8	8	3
8	8	6	4
⑧	8	8	5

g

1	1	1
1	②	1
1	1	1

f

*	*	*	*
*	5	1	*
*	5	2	*
⊛	*	*	*

$g \ominus f$

6	6	6	2
6	6	6	3
6	6	6	3
⑥	6	6	3

$g \circ f$

9	9	9	8	8	3
9	10	9	9	9	4
9	10	10	10	9	5
9	10	10	9	9	6
9	⑩	10	10	9	6
9	9	9	9	9	6

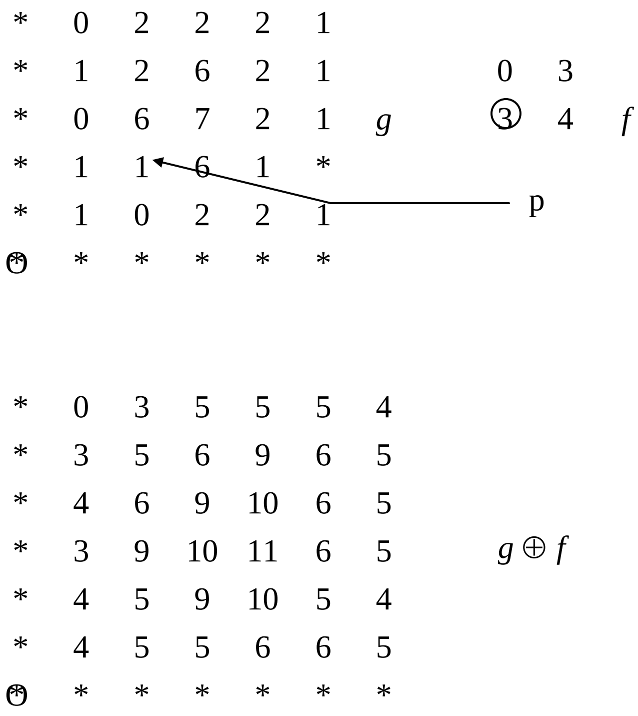
$g \oplus f$

8	7	7	2
8	8	8	3
8	8	8	4
⑧	8	8	5

$g \bullet f$

Figure 19. Example Dilation Calculation

The next example shows a non-symmetrical structuring function [Dougherty, 1992, pp. 115].



*	0	2	2	2	1
*	1	2	6	2	1
*	0	6	7	2	1
*	1	1	6	1	*
*	1	0	2	2	1
O	*	*	*	*	*

0	3
③	4

*	0	3	5	5	5	4
*	3	5	6	9	6	5
*	4	6	9	10	6	5
*	3	9	10	11	6	5
*	4	5	9	10	5	4
*	4	5	5	6	6	5
O	*	*	*	*	*	*

Figure 20. Example Dilation Calculation

The following is an example calculation for pixel p.
Note that the structuring element is effectively reflected about the origin in the calculation.
$\max\{g(p-u)+f(u) \mid u \in Dom[f]\}$ is
max{3+1, 3+1, 0+0, 4+1}=5
Hence $(g \oplus f)(p) = 5$.