1.1 Filter T

Consider now the general case of a filter or transformation T Operating on image g. Then T is translation invariant if $T(g_p + a) = T(g)_p + a$. This says that morphological translation and then filtering is equivalent to filtering and then translating. The operations of erosion and dilation are translation invariant [Dougherty ,1992, pp. 105, 111].

$$(g_p + a)\Theta\, f \;= (\, g\Theta\, f\,)_p + \; a$$

$$\left(g_p + a\right) \oplus f = \left(g \oplus f\right)_p + a$$

$$g \oplus \left(f_p + a\right) = \left(g \oplus f\right)_p + a$$

$$g\,\Theta \left(f_p + a\right) = \left(g\,\Theta f\right)_{-p} - a$$

Filter *T* is antiextensive if T(g)<<g or T(g) is beneath g. Filter T is indempotent if T(T(g))=g. Repeated application gives the same result. The open operator has the following properties.

$(g_p + a) \circ f \;= (g \circ f\,)_p + a$ translation invariant

$(g \circ f\,) \circ f \;= g \circ f$ indempotent

$g \circ f \ll g$ antiextensive

1.2 Morphological Gradient

The morphological gradient is defined by $MG(g) = (g \oplus f\,) - (g\Theta \mathrm{f}\,)$ [Dougherty, 1992, pp. 105, 111]. Recall $g \oplus f \gg g$ and $g\Theta\ \mathrm{f} \ll \mathrm{g}$. For flat structuring elements this gives the difference between the maximum and minimum over the neighborhood of *f* translated to the pixel. Consider the following example.

example

8	7	7	2
8	8	8	3
8	8	6	4
8	8	8	5

g

0	0	0
0	0	0
0	0	0

*	*	*	*
*	6	2	*
*	6	3	*
*	*	*	*

$g \Theta f$

8	8	8	8	7	2
8	8	8	8	8	3
8	8	8	8	8	4
8	8	8	8	8	5
8	8	8	8	8	5
8	8	8	8	8	5

$g \oplus f$

*	*	*	*
*	2	6	*
*	2	5	*
*	*	*	*

$(g \oplus f) - (g \Theta f)$

gradient

Figure 1. Morphological Gradient

1.3 Top-Hat Transform

The top hat transform is defined by tophat $\text{tophat}(g) = \text{g-}(\text{g} \circ \text{f})$ [Dougherty, 1992]. Recall that $g \circ f$ is beneath g or $g \circ f \ll g$. This function is useful for finding pixel groups that are peaks above the background. If bright objects have large gray-levels then the transform would locate bright objects. The Dom[f] should be larger in area than the area of the peaks to be detected. The tophat transform has been used in machine vision (industrial applications) to locate defects and medical applications to locate long thin objects such as blood vessels in an image [Vision Systems Design, June 99]. This function allows one to extract objects with defined shape and contrast characteristics from the background. The following figure indicates the operation. Note that the operation is a generaliztion of the thresholding concept.

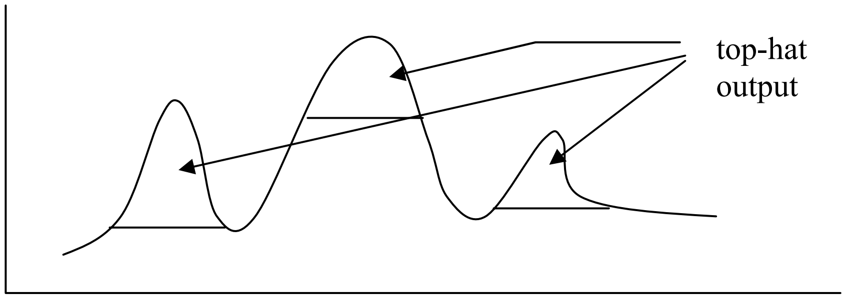

Figure 2. Top-Hat Transform

Consider the following example.

$$\begin{array}{cccc} 8 & 7 & 7 & 2 \\ 8 & 8 & 8 & 3 \\ 8 & 8 & 6 & 4 \\ \textcircled{8} & 8 & 8 & 5 \end{array} \qquad g \qquad\qquad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \textcircled{0} & 0 \\ 0 & 0 & 0 \end{array} \qquad f$$

$$\begin{array}{cccc} 6 & 6 & 6 & 2 \\ 6 & 6 & 6 & 3 \\ 6 & 6 & 6 & 3 \\ \textcircled{6} & 6 & 6 & 3 \end{array} \qquad g \circ f$$

$$\begin{array}{cccc} 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ \textcircled{2} & 2 & 2 & 2 \end{array} \qquad g - \left(g \circ f \right)$$

Figure 3. Top-Hat Example

Example

4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	8	8	4	4	4	4	4
4	4	9	8	4	1	4	4	4
4	4	4	4	4	1	4	4	4
4	4	3	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4

g

0	0	0
0	⓪	0
0	0	0

f

4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
4	4	4	4	4	1	4	4	4
4	4	4	4	4	1	4	4	4
3	3	3	4	4	4	4	4	4
3	3	3	4	4	4	4	4	4
3	3	3	4	4	4	4	4	4

$g \circ f$

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	4	4	0	0	0	0	0
0	0	5	4	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0

$g-(g \circ f)$

Figure 4. Top-Hat Example

1.4 Valley Detector.

Now consider problem of locating vallies in an image. The operator $valley(g) = (g \bullet f) - g$ will detect valleys [Dougherty, 1992, pp. 120]. Recall that $g << g \bullet f$, hence the result is always positive. If dark spots correspond to low gray-level values, then the valley operator will locate the dark spots in an image. Consider the following example.

$$\begin{array}{cccc} 8 & 7 & 7 & 2 \\ 8 & 8 & 8 & 3 \\ 8 & 8 & 6 & 1 \\ \textcircled{8} & 8 & 8 & 5 \end{array} \qquad g \qquad\qquad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \textcircled{0} & 0 \\ 0 & 0 & 0 \end{array} \qquad f$$

$$\begin{array}{cccc} 8 & 7 & 7 & 2 \\ 8 & 8 & 8 & 3 \\ 8 & 8 & 8 & 1 \\ \textcircled{8} & 8 & 8 & 5 \end{array} \qquad g \bullet f$$

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \textcircled{0} & 0 & 0 & 0 \end{array} \qquad (g \bullet f) - g$$

valley detector

Figure 5. Valley Detector Example

example

$$\begin{array}{cccc} 9 & 9 & 9 & 8 \\ 8 & 2 & 1 & 8 \\ 9 & 1 & 1 & 8 \\ \textcircled{8} & 8 & 8 & 8 \end{array} \qquad g \qquad\qquad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \textcircled{0} & 0 \\ 0 & 0 & 0 \end{array} \qquad f$$

$$\begin{array}{cccccc} 9 & 9 & 9 & 9 & 9 & 8 \\ 9 & 9 & 9 & 9 & 9 & 8 \\ 9 & 9 & 9 & 9 & 9 & 8 \\ 9 & 9 & 9 & 9 & 8 & 8 \\ 9 & \textcircled{9} & 9 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 & 8 \end{array} \qquad g \oplus f$$

$$\begin{array}{cccc} 9 & 9 & 9 & 8 \\ 9 & 8 & 8 & 8 \\ 9 & 8 & 8 & 8 \\ \textcircled{8} & 8 & 8 & 8 \end{array} \qquad g \bullet f$$

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 6 & 7 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad (q \bullet f) - g$$

Figure 6. Valley Detector Example

1.5 Geodesic Dilation.

Recall that binary geodesic dilation is defined as shown in the following equations.

$\delta_T^1(A) = \{p \in T \mid d_T(p,A) \le 1\}$

or

$\delta_T^1(A) = \{(A \oplus B) \bigcap T\}$

$\delta_T^n(A) = \underbrace{\delta_T^1 \circ \delta_T^1 \ldots \circ \delta_T^1}_{\text{n times}}(A)$

The reconstruction of T from A is given in the following equations.

$\rho_g(f) = \bigcup_{n\ge1} \delta_T^n(A)$ where A is a marker set and T is a mask set.

The definition of geodesic dilation and erosion can be extended to grayscale images [Vincent, 1993]. First we give some notation. Let g be a grayscale image. Let T be a threshold of the gray-levels. Then

$T_k(g)=\{p \in \mathrm{Dom}(g) \mid g(p) \ge k\}$. Also we note that

$T_k(g) \subseteq T_{k-1}(g)$.

Now consider geodesic dilation. Let

$f \le g$. Then the geodesic dilation of function f with respect to g is

$\delta_g(f) = (f \oplus f_B) \wedge g$. The function g is referred to as the mask and the function f is the marker function. The structuring element f_B is the unit ball. And $\delta_g^2(f) = \delta_g(\delta_g(f))$. The repeated application of the function n times is

$\delta_g^n(f) = \underbrace{\delta_g^1 \circ \delta_g^1 \circ \ldots \circ \delta_g^1}_{\text{n times}}(f)$.

The reconstruction function of g from f is given by the formula

$\rho_g(f) = \bigvee_{n\ge1} \delta_g^n(f)$.

One iterates grayscale geodesic dilation of f under g until stability is reached.

One can define similar functions for geodesic erosion. The equations are given below.

Let $f \ge g$, the geodetic erosion is

$\in_g^1(f) = (f \Theta f_s) \vee g$. The iteration is

$\in_g^n(f) = \underbrace{\in_g^1 \circ \in_g^1 \circ \ldots \circ \in_g^1}_{\text{n times}}(f)$, The reconstruction function is

$\rho_g(f) = \bigwedge_{n\ge1} \in_g^n(f)$. Again one iterates grayscale geodesic erosions of f above g until stability is reached.

Consider the following example. The functions g and f are in the rows of the following table as are the results of repeated application of geodesic dilation.

g	7	6	8	9	8	4	2
f	4	4	4	2	4	2	1
$f \oplus f_s$	5	5	5	5	5	5	3
$\delta_g^1(f)$	5	5	5	5	5	4	2
$\delta_g^1(f) \oplus f_s$	6	6	6	6	6	6	5
$\delta_g^2(f)$	6	6	6	6	6	4	2
$\delta_g^2 \oplus f_s$	7	7	7	7	7	7	5
$\delta_g^3(f)$	7	6	7	7	7	4	2
$\delta_g^3 \oplus f_s$	8	8	8	8	8	8	5
$\delta_g^4(f)$	7	6	8	8	8	4	2
$\delta_g^4 \oplus f_s$	8	9	9	9	9	9	5
$\delta_g^5(f)$	7	6	8	9	8	4	2
$\delta_g^5 \oplus f_s$	8	9	9	9	9	9	5
$\delta_g^6(f)$	7	6	8	9	8	4	2

f_B

1	1	1
1	①	1
1	1	1

Used for the unit ball