1.1 Pattern Spectrum Analysis of Gray Scale Images

The concepts of binary spectra extend naturally to gray-scale images [Maragos, 1989]. Let $nf = f \oplus f \oplus \cdots \oplus f$ performed n times. The concept is to filter the image with progressively larger structuring elements. The filters are $g \circ nf$ and $g \bullet nf$. This is multiscale nonlinear filtering. This is called a spectra because nf is a function used to probe g like the exponentials are used in Fourier analysis. In Fourier analysis one would multiply a function g(x) by $e^{-j2\pi wx}$ and compute the area under the curve as the spectral response at frequency w. In the current situation nf is the probing function
The area of g is given by

$$\mathrm{A(g)} = \sum_{\mathrm{p} \in \mathrm{Dom(g)}} \mathrm{g(p)}$$ in the discrete case.

If one probes g with nf and then computes the area of the resultant function then one gets the spectral response at nf. Now $g \circ nf \geq g \circ (n+1) f$ which implies that $A(g \circ (n+1) f) \leq A(g \circ nf)$. The area is a decreasing function. If g and f are understood then one may write
A(0)=A(g), A(1)= $A(g \circ f)$,...,A(n)= $A(g \circ nf)$. Then
$A(1) \geq A(2) \geq \cdots \geq A(i) \geq \cdots \geq A(n) = A(n+1)$. Let $A(\infty) = A(n)$. Then probability functions may be defined as

$$\mathrm{F(k)} = 1 - \left[\frac{\mathrm{A(k)} - \mathrm{A}(\infty)}{\mathrm{A(0)} - \mathrm{A}(\infty)} \right]$$

where $0 \leq F(k) \leq 1$ a cumulative distribution function. Note that $F(0) = 0$ and $F(\infty) = 1$. The first order probability function is defined by

$$\frac{dF}{dk} = ps(k) = F(k+1) - F(k) = \frac{A(k) - A(k+1)}{A(0) - A(\infty)}$$

. This is the normalized area under the curve of the features that pass $g \circ kf$ but do not pass $g \circ (k+1) f$. These are the features of size kf.
Consider the following example

	9	9	9	9	9	9					
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9					
	9	9	9	9	9	9					
								7			

g

0	0	0
0	0	0
0	0	0

f

	9	9	9	9	9	9					
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9		8	8	8	
	9	9	9	9	9	9					
	9	9	9	9	9	9					

$g \circ f$

g

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

2f

	9	9	9	9	9	9					
	9	9	9	9	9	9					
	9	9	9	9	9	9					
	9	9	9	9	9	9					
	9	9	9	9	9	9					
	9	9	9	9	9	9					

$g \circ 2f$

g

Figure 1. Example Pattern Spectrum Calculation

The result for $g \circ 3f$ consists of all zeros. In this case
A(0)=36(9) + 9*8+7=403
A(1)=36*9+9*8=396
A(2)=36*9=324
A(3)=0
A(4)=0 etc.

$F(0)=0$
$F(1)=1-(396/403)=7/403$
$F(2)=1-(324/403)=79/403$
$F(3)=1-(0/403)=1$
$ps(0)=7/403$
$ps(1)=(79-7)/403=72/403$
$ps(2)=1-(79/403)=324/403$
$ps(3)=0$
$ps(4)=0$
ps(k) is the normalized area under the curve of the features of size kf.

1.2 Roughness.

The following is a measure of the roughness of an image [Maragos, 1989]. $H(g/f) = -\sum_{k=0}^{n} ps(k)\log ps(k)$ where $A(n) = A(\infty)$. This measure is called entropy. It is a maximum when p(k) is uniform. It is a minimum when all the value is concentrated in one k value. This is a measure of the average roughness. Recall the ps(k) is the normalized area under the function g of features of size kf. The features which pass are of size kf. These are the features of g of size kf that protrude above $g \circ kf$. The larger ps(k) then the more features of size kf that protrude above g. When ps(k) = 0 there are no features of size kf. If the only features that protrude above g are of size k only then the ps(k)=1 at that k and zero elsewhere. This says that H is a minimum. If for every k there is an equal volume protruding above g then ps(k) = constant and H is maximum. This would be considered as rougher. For our previous example calculation ps(0)=7/403, ps(1)=72/403, ps(2)=324/403, ps(3)=0 hence

$$H(g/f) = -\sum_{k=0}^{n} ps(k)\log ps(k) = -\left[\frac{7}{403}\log(\frac{7}{403}) + \frac{72}{403}\log(\frac{72}{403}) + \frac{324}{403}\log(\frac{324}{403})\right] =$$

[.306+.127+.0796] = .513.

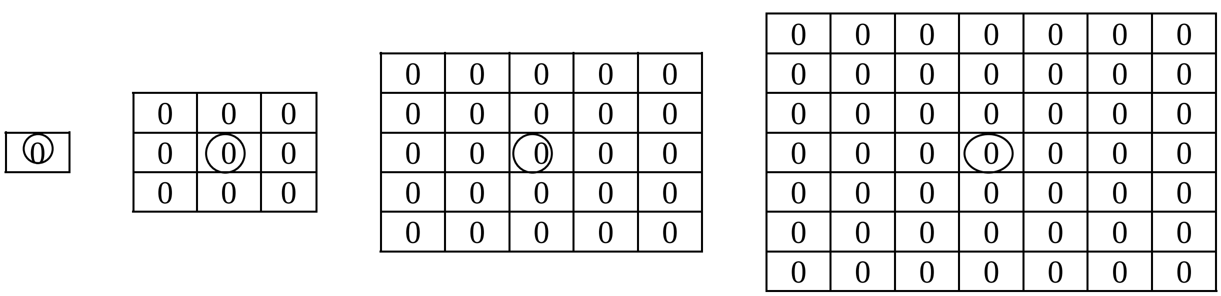
1.3 Foreground/background measurements in a Window.

In a previous section it was seen that for pattern spectrum analysis of binary images, sum(dA) could be used as a measure to estimate the area of foreground/background. Now the same idea is extended to gray scale images. If A_w is the area of the window then one can obtain the percentage area of objects by computing $dA = ps(k) * A(0) = A(k) - A(k+1)$ that gives the area of the features of size kf . We are measuring the area of size kf part of the foreground of the image. One can divide by A_w to obtain the percentage area of the objects [Sundararaman, 1994]. We should note that in this case we are measuring the area under the curve of the object.

Consider the following figure showing a gray level image. The image is successively opened by the structuring element sequence using the gray level open operation. The structuring elements are an increasing sequence of flat, square structuring elements of value '0'.

0	0	0	0	0	0	0
0	5	5	5	5	5	0
0	5	6	6	6	5	0
0	5	6	6	6	5	0
0	5	6	6	6	5	0
0	5	5	5	5	5	0
0	0	0	0	0	0	0

Figure 2. Example Image



◯ indicates the origin

Figure 3. Sequence of Structuring Elements

The following tables give the results of applying the structuring elements. The V difference under the 3x3 column says that the volume for the 3x3 elements is 9. The V difference under the 5x5 element says that the volume for the 5x5 elements is 125. This indicates that this procedure could be employed to determine the foreground volume and hence the foreground percentage over an image.

Table 1. Count for the Opening with the Structuring Element

	Increasing Structuring elements			
	1 **1x1**	**2** **3x3**	**3** **5x5**	**4** **7x7**
V	134	134	125	0

Table 2. Count, ps(n), for the opening with the Structuring Elements

	1	**2**	**3**	**∑dA**
V Difference	0	9	125	134

In the next example, the same procedure is extended to measuring the foreground volume over a window. The gray level image is subjected to opening with the same set of structuring elements, and at each stage the count is taken over a 3x3 window around the pixel. Then the count difference for successive openings, dA, is taken at each pixel. The values of sum(dA) are shown. It can be verified that it is a true representation of the volume of the foreground volume over a 3x3 window at each pixel. From the value of sum(dA), the percentage of foreground (and hence the percentage of background) over the window can then be determined.

3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	6	6	6	6	6	6	6	6	6	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3

Figure 4. Example Image

0	0	0	0	0	0	0	0	0	0	0	0	0
0	3	6	9	9	9	9	9	9	9	6	3	0
0	6	12	18	18	18	18	18	18	18	12	6	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	9	18	27	27	27	27	27	27	27	18	9	0
0	6	12	18	18	18	18	18	18	18	12	6	0
0	3	6	9	9	9	9	9	9	9	6	3	0
0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 5. The Values of dV over a 3x3 Structuring Element.

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