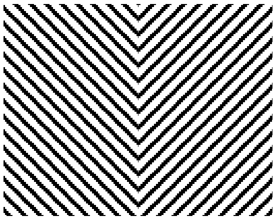
1. First order statistics and Texture

It is possible for textures to differ in first order probabilities and be visually distinguishable. It is also possible for textures to have identical first order statistics and be visually distinct. The following image is an image of four different textural patterns using lines in various orientations. This example demonstrates the distributed nature of the patterns. It also demonstrates that rotation is a factor and that first order statistics are not sufficient to determine texture patterns.



1.1 First Order Statistics

Assume g(x,y) is a random variable with a probability of having a certain value g(x,y)=k. Then p(k)=P{g=k} is the probability that g has value k. If the gray-level values are between 0, 1, ..., L-1, then $\sum_{k=0}^{L-1} P\{g=k\} = \sum_{k=0}^{L-1} p(k) = 1$. This is the first-order probability function. This is also called the histogram h(k) when one estimates p(k) from available image data. The cumulative probability function is denoted by F(k). It is related by the following equations to p(k) the probability function. In the continuous case $F(k) = \int_{-\infty}^{k} p(k)dk$ and $p(k) = \frac{dF}{dk}$. In the discrete case, $F(k) = \sum_{i=0}^{k} p(i)$. F(k) is the probability that $g(x,y) = i \le k$ or $f(k) = P\{g \le k\}$.

The set of all the random variables over the pixels of the image forms a random process. A random process is stationary if the statistics at each pixel are the same [Papoulis, 1965, pp.300]. A random process is ergodic if time averages are the same as ensemble averages [Papoulis, 1965, pp.327]. If one assumes that the random variable is ergodic then the spatial computations over (x,y) may be used to compute F(k) and p(k). If there are L gray-levels then L parameters must be estimated to determine p(i). For this reason one often extracts measures from p(i).

The following example shows two images with identical first order statistics that are visually distinct.

0	0	0	0	0	0	0	0
0	1	1	0	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	0	0	1	0	0
0	1	1	0	0	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	0
0	1	0	1	0	1	0	0
0	1	0	1	0	1	0	0
0	1	0	1	0	1	0	0
0	1	0	1	0	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Figure 1. Visually distinct images

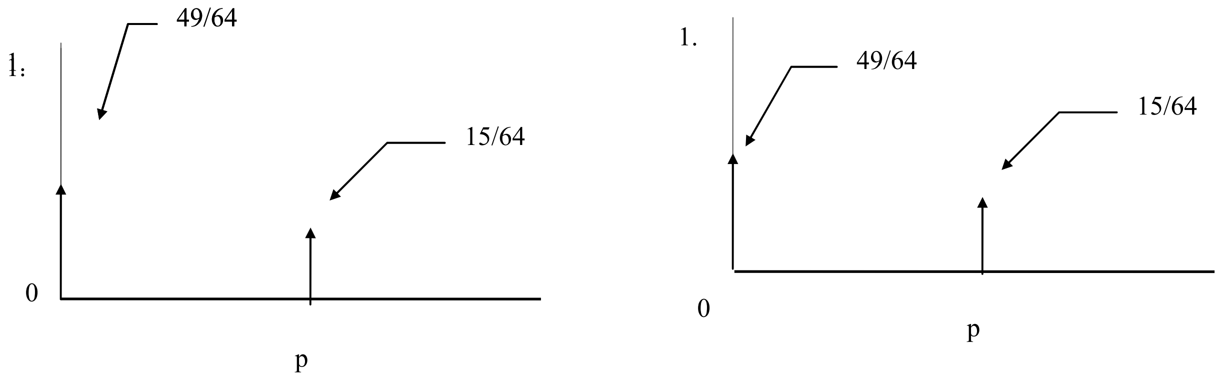


Figure 2. Probability functions

1.2 Measures on the first order probability function

Now consider some measures used to characterize 1st order probability functions [Pratt, 1991, pp. 561; Conners and Harlow, 1980; Haralick, 1986, Gonzales and Woods, 2002, pp. 666]. The mean is given by

$$\mu = \int_{-\infty}^{\infty} \mathrm{kp(k)dk} \text{ for continuous p(k) and } \mu = \sum_{\mathrm{k}} \mathrm{kp(k)}$$

for discrete p(k). The variance is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (k-\mu)^2 p(k)dk \text{ for the continuous case and } \sigma^2 = \sum_{\mathrm{k}} (\mathrm{k}-\mu)^2 \mathrm{p(k)}$$

for the discrete case. The standard deviation is σ which measures the dispersion of the function about the mean. The larger σ then the wider the function is dispersed.

The moment and central moment equations are given below.

$$\mu_i = \int_{-\infty}^{\infty} \mathrm{k}^i \mathrm{p(k)dk} \text{ in the continuous case or } \mu_i = \sum_k k^i p(k) \text{ in the discrete case and}$$

$$\overline{\mu}_i = \int_{-\infty}^{\infty} (k-\mu)^i p(k)dk \text{ in the continuous case or } \overline{\mu}_i = \sum_k (\mathrm{k}-\mu)^i \mathrm{p(k)} \text{ in the continuous case. These}$$

quantities are the ith moment and the ith central moment.

The median or central value is given by $\tilde{k}$ which is defined by $P\{k < \tilde{k}\} = .5$. The skewness is defined by

$\text{skewness}=\frac{\overline{\mu}_3}{\overline{\mu}_2^{\frac{3}{2}}}=\frac{\overline{\mu}_3}{\sigma^3}$. One may also use $\overline{\mu}_3$ to measure skewness [Gonzalez and Woods, 2002, pp. 666].

The kurtosis is defined by

$\text{kurtosis}=\frac{\overline{\mu}_4}{\overline{\mu}_2^2}\text{-3}=\frac{\overline{\mu}_4}{\sigma^4}\text{-3}$. The 3 is subtracted from kurtosis so that the kurtosis of a Gaussian probability function is 0 [Levine, 1985,pp. 449].

The entropy is defined by

$\text{entropy} = -\sum_{k=0}^{L-1} p(k)\log(p(k))$. The entropy measure is 0 for a constant image and maximum for an image with a uniform histogram.

A symmetry measure is defined by

$$\text{symmetry} = \frac{-\sum_{\mathrm{k}=0}^{\hat{\mathrm{k}}} \mathrm{p(k)log(p(k))}}{\text{entropy}} \quad \text{where } \hat{\mathrm{k}}=\min\{\mathrm{k} \mid \mathrm{F(k)} \geq .5\}$$

A symmetric histogram with mirror axis at L/2 will have a symmetry value of .5. The entropy measure is zero for an image with a constant gray-level. The variance measures the dispersion of the p(k). The skewness is an odd function and will give large values for asymmetrical p(k). The kurtosis will give large values for p(k) which have outliers or large values at high values of k.

The energy is $E = \sum_{k=0}^{L-1} p(k)^2$. This is also a measure of uniformity [Gonzalez and Woods, 2002, pp. 666].

The mean reflects the average gray-level. The variance measures the dispersion of the gray-levels. The skewness measures the departure from symmetry. The symmetry measure measures the symmetric of p(k). The kurtosis measures the outliers or the tendency of the distribution to spread towards the tails.

Consider the following example which gives calculations of these quantities.

	3	2	2	1	1	1	1	1
	5	3	2	2	1	1	1	1
	7	6	3	2	2	1	1	1
y	7	7	5	2	1	1	1	1
	8	7	7	2	2	1	1	1
	8	8	8	3	2	1	1	1
	8	8	6	4	2	1	2	2
	8	8	8	5	1	1	1	1

x

Image

g(x,y)

The probability functions are given by the following table.

k	p(k)	F(k)
1	.421875	.421875
2	,203125	.625
3	.0625	.6875
4	.015625	.703125
5	.046875	.75
6	.03125	.78125
7	.078125	.859375
8	.140625	1

$\mu=3.17187$ $\sigma^2=7.11108$ $\sigma=2.66699$
kurtosis =-.91328 skewness=.873426

1.3 References

Conners, R. W., and Harlow, C. A. (1980). “Toward a Structural Textural Analyzer Based on Statistical Methods.” *Computer Graphics and Image Processing*, Vol. 12m 1980, pp. 224-256.

Haralick, R. M. (1986). “Statistical Image Texture Analysis.” Handbook of Pattern Recognition and Image Processing, Academic Press, pp. 247-279.

Levine, M. D. (1985). *Vision in Man and Machine*, McGraw-Hill Book Company.

Papoulis, A. (1965). *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill Book Company.

Pratt, W. K. (1991). *Digital Image Processing*, John Wiley & Sons.